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Resonance Problems for Nonlinear Elliptic Equations with Nonlinear Boundary Conditions

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Abstract

We study the solvability of nonlinear second order elliptic partial differential equations with nonlinear boundary conditions where we impose asymptotic conditions on both nonlinearities in the differential equation and on the boundary in such a way that resonance occurs at a generalized eigenvalue; which is an eigenvalue of the linear problem in which the spectral parameter is both in the differential equation and on the boundary. The proofs are based on some variational techniques and topological degree arguments.

Keywords: nonlinear elliptic equations, nonlinear boundary conditions, weighted Robin-Neumann-Steklov eigenproblem, resonance conditions.

1 Introduction

In this paper we prove the existence of (weak) solutions to nonlinear second order elliptic partial differential equations with (possibly) nonlinear boundary conditions

$$\begin{cases} -\Delta u + c(x)u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the nonlinear reaction-function $f(x,u)$ and the nonlinear boundary function $g(x,u)$ interact, in some sense, with the generalized spectrum of the following linear problem (with possibly singular (m,ρ)-weights)

$$\begin{cases} -\Delta u + c(x)u = \mu m(x)u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu\rho(x)u & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Notice that the eigenproblem (2) includes as special cases the weighted Steklov eigenproblem (when $m \equiv 0$ and $\rho \not\equiv 0$) which was considered in [3, 4, 6, 17, 24] as well as the weighted Robin-Neumann eigenproblem (when $\rho \equiv 0$ and $m \not\equiv 0$); the latter is also referred to in the literature as Neumann or regular oblique derivative boundary condition (see e.g. [1, 16] and references therein). When $m \not\equiv 0$ and $\rho \not\equiv 0$, we have then the eigenparameter μ both in the differential equation and on the boundary condition, we refer for instance to [5, 7, 8, 18].
Unlike previous results in the literature, what sets our results apart is that we compare both the reaction nonlinearity f in the differential and the boundary nonlinearity g in Eq.(1) with *higher* eigenvalues of the spectrum of problem (2), where the spectral parameter is both in the differential equation and on the boundary (with weights).
The nonlinear problem (1) has received much attention in recent years. Such problem (and its parabolic analog) has been studied in [9, 22] as a model for heat conduction in a body where cooling and heating appear inside and at the boundary at a rate proportional to a power of u. Problem (1) has also been considerably studied by many authors in the framework of sub and super-solutions method. We refer e.g. to [1, 2, 21], and references therein. Since it is based on (the so-called) comparison techniques, the (ordered) sub-super solutions method does not apply when the nonlinearities are compared with higher eigenvalues.
After Landesman-Lazer [14], much work has been devoted to the study of the solvability of elliptic boundary value problems (with linear homogeneous boundary conditions) where the reaction nonlinearity in the differential equation interacts with the eigenvalues of the corresponding linear differential equation with linear homogeneous boundary conditions (resonance and nonresonance problems). For some recent results in this direction we refer e.g. to [11, 12, 13, 19, 20, 23], and references therein.
A few results on a disk ($n = 2$) were obtained in the case of linear elliptic equations with nonlinear boundary conditions, where the nonlinearity on the boundary was compared with the first Steklov eigenvalue (that is, $m \equiv 0$ in Eq.(2)). We refer to Cushing [10] and Klingelhöfer [15] (the results in [15] were significantly generalized to higher dimensions in [1] in the framework of sub and super-solutions method as aforementioned). In [3, 17] the resonance problem for elliptic equations with nonlinear boundary conditions was analyzed using bifurcation theory (see Remark 3.6 herein). More recently, the authors in [19] proved nonresonance results for problem (1) in which the nonlinearities interact, in some sense, only with either the Steklov or the Neumann spectrum. In a very recent paper of one of the authors [18], *nonresonance* results for problem (1) were proved in which both nonlinearities in the differential equation and on the boundary interact, in some sense, with the generalized spectrum of problem (2).
It is our purpose in this paper to establish existence results for problem (1) by imposing asymptotic conditions on both nonlinearities in the differential equation and on the boundary in such a way that *resonance* occurs at a generalized eigenvalue of problem (2). Our results generalize earlier ones in [1, 2, 12, 23], and in some instances some of those in [3, 11, 17] which were obtained in a different setting.
The content of this paper is organized as follows. In Section 2, we state and prove some preliminary results that we shall need in the sequel. Section 3 is devoted to the statement and proof of existence results for Eq.(1) at *resonance*. The proof of the main result is based on variational and topological degree arguments. We conclude the paper with some remarks which show (among other) how our result can be extended to problems with variable

coefficients.
Throughout this paper we assume that Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with boundary $\partial\Omega$ of class $C^{0,1}$, $\partial/\partial\nu$ is the (unit) outer normal derivative. By a weak solution of Eq.(1) we mean a function $u \in H^1(\Omega)$ such that

$$\int \nabla u \nabla v + \int c(x)uv + \oint \sigma(x)uv = \int f(x,u)v + \oint g(x,u)v \quad \text{for all } v \in H^1(\Omega), \tag{3}$$

where $\int$ denotes the (volume) integral on Ω and $\oint$ denotes the (surface) integral on $\partial\Omega$.
Let us mention that $H^1(\Omega)$ denotes the usual real Sobolev space of functions on Ω endowed with the (c,σ)-inner product defined by

$$(u,v)_{(c,\sigma)} = \int \nabla u \nabla v + \int c(x)uv + \oint \sigma(x)uv \tag{4}$$

with the associated norm denoted by $\|u\|_{(c,\sigma)}$. (The conditions on c and σ which imply that (4) is an inner product are given below.) This norm is equivalent to the standard $H^1(\Omega)$-norm. Besides the Sobolev spaces, we shall make use, in what follows, of the real Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq \infty$, and the compactness of the trace operator $\Gamma : H^1(\Omega) \to L^q(\partial\Omega)$ for $1 \leq q < \dfrac{2(n-1)}{n-2}$. (Sometimes we will just use u in place of Γu when considering the trace of a function on $\partial\Omega$.)
The functions $c : \Omega \to \mathbb{R}$, $\sigma : \partial\Omega \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.

(C1) $c \in L^p(\Omega)$ with $p \geq n/2$ when $n \geq 3$ ($p > 1$ when $n = 2$) and $\sigma \in L^q(\partial\Omega)$ with $q \geq n-1$ when $n \geq 3$ ($q > 1$ when $n = 2$) with $(c,\sigma) > 0$; that is, $c(x) \geq 0$ a.e. on Ω and $\sigma(x) \geq 0$ a.e. on $\partial\Omega$ such that

$$\int c(x)\,dx + \oint \sigma(x)\,dx \neq 0.$$

(C2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e., measurable in x for each u, and continuous in u for a.e. x).

(C3) There exist constants a_1, $a_2 > 0$ such that for a.e. $x \in \partial\Omega$ and all $u \in \mathbb{R}$,

$$|g(x,u)| \leq a_1 + a_2|u|^s \quad \text{with } 0 \leq s < \frac{n}{n-2}.$$

(C3') There exist constants b_1, $b_2 > 0$ such that for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$,

$$|f(x,u)| \leq b_1 + b_2|u|^s \quad \text{with } 0 \leq s < \frac{n+2}{n-2}.$$

2 Generalized Eigenproblems and Weighted Nonresonance

To put our results into context, we have collected in this short section some relevant results on generalized linear eigenproblems and *nonresonance* for nonlinear elliptic problem (1) needed for our purposes. We refer to a paper of one of the authors [18] for the proofs of these results.

Consider the linear problem

$$\begin{cases} -\Delta u + c(x)u = \mu m(x)u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu\rho(x)u & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $(m, \rho) \in L^p(\Omega) \times L^q(\partial\Omega)$ with p and q as in Section 1, and $(m, \rho) > 0$; that is,

$$m(x) \geq 0 \text{ a.e. on } \Omega \text{ and } \rho(x) \geq 0 \text{ a.e. on } \partial\Omega \text{ with } \int m(x)\,dx + \oint \rho(x)\,dx \neq 0. \tag{6}$$

(We stress the fact that the weight-functions m and ρ may vanish on subsets of positive measure.)
The (generalized) eigenproblem is to find a pair $(\mu, \varphi) \in \mathbb{R} \times H^1(\Omega)$ with $\varphi \not\equiv 0$ such that

$$\int \nabla\varphi\nabla v + \int c(x)\varphi v + \oint \sigma(x)\varphi v = \mu\left(\int m(x)\varphi v + \oint \rho(x)\varphi v\right) \text{ for all } v \in H^1(\Omega). \tag{7}$$

Picking $v = \varphi$ it follows that, if there is such an eigenpair, then one has that $\mu > 0$ and $\int m(x)\varphi^2 + \oint \rho(x)\varphi^2 > 0$. Therefore, one can split the Hilbert space $H^1(\Omega)$ as a direct (c, σ)-orthogonal sum in the following way,

$$H^1(\Omega) = V_{(m,\rho)}(\Omega) \oplus H^1_{(m,\rho)}(\Omega), \tag{8}$$

where $V_{(m,\rho)}(\Omega) := \left\{u \in H^1(\Omega) : \int m(x)u^2 + \oint \rho(x)u^2 = 0\right\}$ and $H^1_{(m,\rho)}(\Omega) = \left[V_{(m,\rho)}(\Omega)\right]^{\perp}$.
On $H^1_{(m,\rho)}(\Omega) \subset H^1(\Omega)$, we will also consider the inner product defined by

$$(u, v)_{(m,\rho)} := \int m(x)uv + \oint \rho(x)uv, \tag{9}$$

with corresponding norm denoted by $||\cdot||_{(m,\rho)}$ (see e.g. [18] for details).
Assuming that the above assumptions are satisfied, one of the authors [18] proved that, for $n \geq 2$, the eigenproblem (5) has a sequence of real eigenvalues

$$0 < \mu_1 < \mu_2 \leq \ldots \leq \mu_j \leq \ldots \to \infty, \text{ as } j \to \infty,$$

each eigenvalue has a finite-dimensional eigenspace. The eigenfunctions φ_j corresponding to these eigenvalues form a complete orthonormal family in the (proper) subspace $H^1_{(m,\rho)}(\Omega)$. Moreover, the first eigenvalue μ_1 is simple, and its associated eigenfunction φ_1 is strictly positive (or strictly negative) in Ω and the following inequalities hold.

(i) For all $u \in H^1(\Omega)$,

$$\mu_1\left(\int m(x)u^2 + \oint \rho(x)u^2\right) \leq \int |\nabla u|^2 + \int c(x)u^2 + \oint \sigma(x)u^2, \tag{10}$$

where $\mu_1 > 0$ is the least eigenvalue for Eq.(5). If equality holds in (10), then u is a multiple of an eigenfunction of Eq.(5) corresponding to μ_1.

(ii) For every $v \in \oplus_{i\leq j}E(\mu_i)$, and $w \in \oplus_{i\geq j+1}E(\mu_i)$, we have that

$$||v||^2_{(c,\sigma)} \leq \mu_j||v||^2_{(m,\rho)} \qquad \textit{and} \qquad ||w||^2_{(c,\sigma)} \geq \mu_{j+1}||w||^2_{(m,\rho)}, \tag{11}$$

where $E(\mu_i)$ is the μ_i-eigenspace, $\oplus_{i\leq j}E(\mu_i)$ is the span of eigenfunctions associated with eigenvalues below and up to μ_j, and $||\cdot||_{(m,\rho)}$ is the norm induced by (9).

The next theorem concerns an existence result for the nonlinear problem (1) in the case of *weighted nonresonance*. We refer to [18] for the proof of this result.

Theorem 2.1 (Weighted nonresonance between consecutive generalized eigenvalues)
Suppose that the assumptions (C1)-(C3') and (6) are met, and that the following conditions hold.
(C4) There exist constants a, b, α, $\beta \in \mathbb{R}$ *such that*

$$\alpha m(x) \leq \liminf_{|u|\to\infty} \frac{f(x,u)}{u} \leq \limsup_{|u|\to\infty} \frac{f(x,u)}{u} \leq \beta m(x)$$

and

$$a\rho(x) \leq \liminf_{|u|\to\infty} \frac{g(x,u)}{u} \leq \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq b\rho(x),$$

uniformly for a.e. $x \in \Omega$*, respectively for a.e.* $x \in \partial\Omega$*, where*

$$\mu_j < \min(a,\alpha) \leq \max(b,\beta) < \mu_{j+1}. \tag{12}$$

Then, Eq.(1) has at least one (weak) solution $u \in H^1(\Omega)$*.*

Remark 2.2 The result in Theorem 2.1 remains valid if one replaces the functions $f(x,u)$ and $g(x,u)$ with $f(x,u) + A(x)$ and $g(x,u) + B(x)$ respectively, where $A \in L^2(\Omega)$ and $B \in L^2(\partial\Omega)$.

3 Main Result

In this section, we prove an existence result for problem (1) at *resonance* which includes both the Steklov as well as Neumann and Robin problems with appropriate choices of the weights m and ρ as aforementioned. Notice that the nonlinearity in the boundary condition is at resonance as well. We require that the nonlinearities satisfy some sign conditions and a Landesman-Lazer type condition (possibly at a generalized higher eigenvalue).
Consider the following nonlinear problem

$$\begin{cases} -\Delta u + c(x)u = \mu_j m(x)u + f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu_j\rho(x)u + g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where μ_j is a generalized eigenvalue of the (weighted) problem (5).

Theorem 3.1 *(Resonance at any generalized eigenvalue)*
Suppose that the assumptions (C1)–(C3') and (6) are met, and that the following conditions hold.

(C5) There exists a constant β such that

$$0 \leq \liminf_{|u|\to\infty} \frac{f(x,u)}{u} \leq \limsup_{|u|\to\infty} \frac{f(x,u)}{u} \leq \beta m(x)$$

and

$$0 \leq \liminf_{|u|\to\infty} \frac{g(x,u)}{u} \leq \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq \beta \rho(x),$$

uniformly for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, where $\beta < (\mu_{j+1} - \mu_j)$.

(C6) Sign conditions: There exist functions $a, A \in L^2(\Omega)$, $b, B \in L^2(\partial\Omega)$, and constants $r < 0 < R$ such that

$$f(x,u) \geq A(x) \quad \textit{and} \quad g(x,u) \geq B(x)$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$ and all $u \geq R$,

$$f(x,u) \leq a(x) \qquad \textit{and} \qquad g(x,u) \leq b(x)$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$ and all $u \leq r$.

Then, Eq. (13) has at least one (weak) solution $u \in H^1(\Omega)$ provided that the following Landesman-Lazer type condition holds:

$$\int_{\varphi>0} f_+\varphi + \int_{\varphi<0} f_-\varphi + \oint_{\varphi>0} g_+\varphi + \oint_{\varphi<0} g_-\varphi > 0 \quad \textit{for all } \varphi \in E(\mu_j) \setminus \{0\}, \tag{14}$$

where $E(\mu_j)$ is the μ_j-eigenspace, $f_+(x) := \liminf_{u\to\infty} f(x,u)$, $g_+(x) := \liminf_{u\to\infty} g(x,u)$, $f_-(x) := \limsup_{u\to-\infty} f(x,u)$, $g_-(x) := \limsup_{u\to-\infty} g(x,u)$ and, $\int_{\varphi>0}$ and $\oint_{\varphi>0}$ denote the integrals on the sets $\{x \in \Omega : \varphi(x) > 0\}$ and $\{x \in \partial\Omega : \varphi(x) > 0\}$ respectively.

Unlike previous results in the literature, what sets our results apart is that we compare both the reaction nonlinearity f in the differential equation and the boundary nonlinearity g with *higher* eigenvalues of the spectrum of problem (2), where the spectral parameter is both in the differential equation and on the boundary (with possibly singular weights). It should also be noted that the presence of the boundary nonlinearity extends the range of allowable 'forcing' terms in the condition (14). Our results generalize earlier ones in [1, 2, 3, 12, 17, 23] (see Remark 3.6 for details).

We will use variational and topological degree techniques combined with some duality arguments. Before giving a proof of our main result, we first prove several lemmas that are relevant in order to obtain a priori estimates. (For some of these lemmas, we borrow some techniques of proof from [12, 13].)

For $u \in H^1(\Omega)$, we shall write

$$u = u^0 + \bar{u} + \tilde{u} + v,$$

where $u^0 \in E^0 := E(\mu_j)$, $\bar{u} \in \oplus_{i\leq j-1} E(\mu_i)$, $\tilde{u} \in \oplus_{i\geq j+1} E(\mu_i)$, and $v \in V_{(m,\rho)}$. Moreover, we shall set

$$w := \tilde{u} + v - \bar{u} - u^0 \quad \text{and} \quad u^\perp := \tilde{u} + v + \bar{u}.$$

Lemma 3.2 *Let $\beta > 0$ be as in Theorem 3.1. Then there exists $\delta = \delta(\beta) > 0$ such that for all $u \in H^1(\Omega)$*

$$\int \nabla u \nabla w + \int c(x)uw + \oint \sigma(x)uw - (\mu_j + \beta)\left(\int m(x)uw + \oint \rho(x)uw\right) \geq \delta \left\|u^{\perp}\right\|^2_{(c,\sigma)}.$$

Proof. Let $u \in H^1(\Omega)$, define $D_\beta(u)$ by

$$D_\beta(u) := \int \nabla u \nabla w + \int c(x)uw + \oint \sigma(x)uw - (\mu_j + \beta)\left(\int m(x)uw + \oint \rho(x)uw\right).$$

Taking into account the (c,σ)-orthogonality of $\tilde{u}$, v, $\bar{u}$ and u^0 in $H^1(\Omega)$ and the fact that $v \in V_{(m,\rho)}$ and $u^0 \in E^0$, one has that

$$\begin{aligned} D_\beta(u) &= \|\tilde{u}\|^2_{(c,\sigma)} + \|v\|^2_{(c,\sigma)} - \|\bar{u}\|^2_{(c,\sigma)} - (\mu_j + \beta)\|\tilde{u}\|^2_{(m,\rho)} + (\mu_j + \beta)\|\bar{u}\|^2_{(m,\rho)} + \beta\left\|u^0\right\|^2_{(m,\rho)} \\ &\geq \left(\|\tilde{u}\|^2_{(c,\sigma)} - \frac{(\mu_j + \beta)}{\mu_{j+1}}\|\tilde{u}\|^2_{(c,\sigma)}\right) + \|v\|^2_{(c,\sigma)} + \left(\frac{\mu_j}{\mu_{j-1}}\|\bar{u}\|^2_{(c,\sigma)} - \|\bar{u}\|^2_{(c,\sigma)}\right) \\ &\geq \delta\left(\|\tilde{u}\|^2_{(c,\sigma)} + \|v\|^2_{(c,\sigma)} + \|\bar{u}\|^2_{(c,\sigma)}\right) = \delta\left\|u^{\perp}\right\|^2_{(c,\sigma)}, \end{aligned}$$

where $\delta = \min\left\{1, 1 - \dfrac{\mu_j + \beta}{\mu_{j+1}}, \dfrac{\mu_j}{\mu_{j-1}} - 1\right\}$. The proof is complete. □

Lemma 3.3 *Let $\beta > 0$ be as in Theorem 3.1, $\delta > 0$ be associated with β by Lemma 3.2, and $\epsilon > 0$. Then for all $\bar{\tau} \in L^p(\Omega)$ and $\tilde{\tau} \in L^q(\partial\Omega)$ satisfying $\bar{\tau}(x) \leq \beta m(x) + \epsilon$, $\tilde{\tau}(x) \leq \beta\rho(x) + \epsilon$ respectively, and all $u \in H^1(\Omega,)$ one has*

$$(u, w)_{(c,\sigma)} - \mu_j\left(\int m(x)uw + \oint \rho(x)uw\right) - \int \bar{\tau}(x)uw + \oint \tilde{\tau}(x)uw \geq C_\delta \left\|u^{\perp}\right\|^2_{(c,\sigma)},$$

where $C_\delta > 0$ is a constant depending on δ and ϵ, provided that $\epsilon > 0$ is sufficiently small.

Proof. Let $D_\tau(u) := (u, w)_{(c,\sigma)} - \mu_j\left(\int m(x)uw + \oint \rho(x)uw\right) - \int \bar{\tau}(x)uw + \oint \tilde{\tau}(x)uw$. Using the computations in the proof of Lemma 3.2 we obtain that

$$D_\tau(u) \geq \delta\left\|u^{\perp}\right\|^2_{(c,\sigma)} - \epsilon\tilde{K}\left\|u^{\perp}\right\|^2_{(c,\sigma)} = (\delta - \epsilon\tilde{K})\left\|u^{\perp}\right\|^2_{(c,\sigma)} = C_\delta\left\|u^{\perp}\right\|^2_{(c,\sigma)},$$

where $\tilde{K}$ is a constant. If ϵ is sufficiently small then we get that $C_\delta > 0$. The proof is complete □

Lemma 3.4 *Assume (C1)–(C3') and (6) are met, and that (in addition) f and g satisfy the sign-condition (C6). Then, for each real number $K > 0$ there are decompositions*

$$f(x,u) = p_K(x,u) + f_K(x,u) \quad \text{and} \quad g(x,u) = q_K(x,u) + g_K(x,u) \tag{15}$$

of f and g such that

$$0 \leq u\,p_K(x,u) \quad \text{and} \quad 0 \leq u\,q_K(x,u) \tag{16}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$. Moreover, there exist functions $\bar{\omega} \in L^2(\Omega)$ and $\tilde{\omega} \in L^2(\partial\Omega)$ depending on a, A, f and b, B, g respectively such that

$$|f_K(x,u)| \leq \bar{\omega}(x) \quad \text{and} \quad |g_K(x,u)| \leq \tilde{\omega}(x) \tag{17}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$.

Proof. Given $K > 0$, define $\hat{g}_K(x,u) := \begin{cases} \inf\{g(x,u), K\} & \text{if } u \geq 1, \\ \sup\{g(x,u), K\} & \text{if } u \leq -1, \end{cases}$
and $\hat{q}_K(x,u) := g(x,u) - \hat{g}_K(x,u)$ for $x \in \overline{\Omega}$ and $|u| \geq 1$.

Set $q_K(x,u) := \begin{cases} \hat{q}_K(x,u) & \text{if } |u| \geq 1, \\ u\,\hat{q}_K(x, u/|u|) & \text{if } 0 < |u| \leq 1, \\ 0 & \text{if } u = 0. \end{cases}$
Finally, define $g_K := g - q_K$. By an easy calculation, one can check that all the conditions of the lemma are satisfied with $\tilde{\omega} = \tilde{C} + \max\{|b(x)|, |B(x)|, K\}$, where the constant $\tilde{C} > 0$ depends on R, $-r$, 1, b_1, b_2. Similar arguments are used in the case of the function f. The proof is complete. □

Lemma 3.5 *Assume (C1)–(C3') and (6) are met, and that in (addition) f and g satisfy (C5) and (C6). Then, for each real number $K > 0$, the functions p_K and q_K provided by Lemma 3.4 satisfy the following additional conditions*

$$|p_K(x,u)| \leq (\beta\, m(x) + \epsilon)|u| - K \quad \text{and} \quad |q_K(x,u)| \leq (\beta\, \rho(x) + \epsilon)|u| - K \tag{18}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \max\{1, \epsilon\}$.

Proof. It follows from (C5) that for all $\epsilon > 0$ there exists $\kappa = \kappa(\epsilon) > 0$ such that

$$|f(x,u)| \leq (\beta\, m(x) + \epsilon)|u| \qquad \text{and} \qquad |g(x,u)| \leq (\beta\, \rho(x) + \epsilon)|u| \tag{19}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and $u \in \mathbb{R}$ with $|u| \geq \kappa$.

Let $u \in \mathbb{R}$ with $|u| \geq 1$. Then $\hat{g}_K(x,u) := \begin{cases} g(x,u) & \text{if } u \geq 1 \text{ and } g(x,u) \leq K, \\ K & \text{if } u \geq 1 \text{ and } g(x,u) \geq K, \\ g(x,u) & \text{if } u \leq -1 \text{ and } g(x,u) \geq -K, \\ K & \text{if } u \leq -1 \text{ and } g(x,u) \leq -K. \end{cases}$

It follows that $q_K(x,u) = \begin{cases} 0 & \text{if } u \geq 1 \text{ and } g(x,u) \leq K, \\ g(x,u) - K & \text{if } u \geq 1 \text{ and } g(x,u) \geq K, \\ 0 & \text{if } u \leq -1 \text{ and } g(x,u) \geq -K, \\ g(x,u) + K & \text{if } u \leq -1 \text{ and } g(x,u) \leq -K. \end{cases}$

By (19) we get that

$$0 \leq q_K(x,u) \leq (\beta\, \rho(x) + \epsilon)\, u - K \quad \text{if } \ u \geq \max\{1, \kappa\}$$

and

$$0 \geq q_K(x,u) \geq (\beta\,\rho(x)+\epsilon)\,u+K \quad \text{if } u \leq -\max\{1,\kappa\}.$$

Therefore

$$|q_K(x,u)| \leq (\beta\,\rho(x)+\epsilon)\,|u|-K$$

for a.e. $x \in \partial\Omega$ and all $u \in \mathbb{R}$ with $|u| \geq \max\{1,\kappa\}$. Similar arguments are used in the case of f. The proof is complete. □

Proof of Theorem 3.1. The proof is divided into four steps.

Step 1. Let δ be associated to the constant β by Lemma 3.2. Then, by assumption (C5), there exists $\kappa \equiv \kappa_\delta > 0$ such that

$$|f(x,u)| \leq (\beta\,m(x)+\tilde{d})|u| \qquad \text{and} \qquad |g(x,u)| \leq (\beta\,\rho(x)+\tilde{d})|u| \tag{20}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \kappa$, where $\tilde{d}$ is a sufficiently small constant such that $0 < \tilde{d} << \delta$. By using Lemma 3.4 with $K = 1$, Eq.(13) is equivalent to

$$\begin{cases} -\Delta u + c(x)u = \mu_j m(x)u + p_1(x,u) + f_1(x,u) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu_j \rho(x)u + q_1(x,u) + g_1(x,u) & \text{on } \partial\Omega, \end{cases} \tag{21}$$

where p_1, f_1, q_1 and g_1 are defined in Lemma 3.4 and satisfy conditions (16) and (17). Moreover, since f and g verify the inequalities (20), by Lemma 3.5 we get that

$$|p_1(x,u)| \leq (\beta\,m(x)+\tilde{d})|u|+1 \qquad \text{and} \qquad |q_1(x,u)| \leq (\beta\,\rho(x)+\tilde{d})|u|+1 \tag{22}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \max\{1,\,\kappa\}$ (see the construction of p_1 and q_1 in Lemma 3.4). Let us choose $\bar{\kappa} \geq \max\{1,\,\kappa\}$ so that $(1/|u|) < \tilde{d}$. It follows that

$$0 \leq \frac{p_1(x,u)}{u} \leq \beta\,m(x)+d \qquad \text{and} \qquad 0 \leq \frac{q_1(x,u)}{u} \leq \beta\,\rho(x)+d, \tag{23}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \bar{\kappa}$, where $d = 2\tilde{d} << \delta$.

Step 2. Let us define $\bar{\gamma}$, $\tilde{\gamma} : \bar{\Omega} \times \mathbb{R}$ by

$$\bar{\gamma}(x,u) = \begin{cases} \dfrac{p_1(x,u)}{u} & \text{for } |u| \geq \bar{\kappa}\\ \dfrac{p_1(x,\bar{\kappa})+p_1(x,-\bar{\kappa})}{2\bar{\kappa}^2}\,u + \dfrac{p_1(x,\bar{\kappa})-p_1(x,-\bar{\kappa})}{2\bar{\kappa}} & \text{for } |u| < \bar{\kappa}. \end{cases}$$

$$\tilde{\gamma}(x,u) = \begin{cases} \dfrac{q_1(x,u)}{u} & \text{for } |u| \geq \bar{\kappa}\\ \dfrac{q_1(x,\bar{\kappa})+q_1(x,-\bar{\kappa})}{2\bar{\kappa}^2}\,u + \dfrac{q_1(x,\bar{\kappa})-q_1(x,-\bar{\kappa})}{2\bar{\kappa}} & \text{for } |u| < \bar{\kappa}. \end{cases}$$

The functions $\bar{\gamma}$ and $\tilde{\gamma}$ are Carathéodory in $\overline{\Omega} \times \mathbb{R}$ since p_1 and q_1 are. Moreover, by (23) one has

$$0 \leq \bar{\gamma}(x,u) \leq \beta\,m(x)+d \qquad \text{and} \qquad 0 \leq \tilde{\gamma}(x,u) \leq \beta\,\rho(x)+d, \tag{24}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$.
Define $\bar{h}, \tilde{h} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{h}(x,u) = f_1(x,u) + p_1(x,u) - \bar{\gamma}(x,u)\,u \qquad \text{and} \qquad \tilde{h}(x,u) = g_1(x,u) + q_1(x,u) - \tilde{\gamma}(x,u)\,u,$$

then it follows from (17) that for some $\bar{\zeta}(x) \in L^2(\Omega)$ and $\tilde{\zeta}(x) \in L^2(\partial\Omega)$,

$$|\bar{h}(x,u)| \le \bar{\zeta}(x) \quad \textit{and} \quad |\tilde{h}(x,u)| \le \tilde{\zeta}(x)$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$, where $\bar{\zeta}$, $\tilde{\zeta}$ depend on β, m, ρ, $\bar{\kappa}$ and the bounds of f_1 and g_1.
Finally, Eq.(21) is equivalent to

$$\begin{cases} -\Delta u + c(x)u = \mu_j m(x)u + \bar{\gamma}(x,u)\,u + \bar{h}(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu_j \rho(x)u + \tilde{\gamma}(x,u)\,u + \tilde{h}(x,u) & \text{on } \partial\Omega. \end{cases} \tag{25}$$

We will use the Leray-Schauder Fixed Point Theorem to prove that Eq.(25) has at least one (weak) solution. In order to apply this theorem, we need to show the existence of an *a priori* bound for all possible (weak) solutions of the family of equations

$$\begin{cases} -\Delta u + c(x)u - \mu_j\, m(x)u - (1-\lambda)d\, m(x))u - \lambda\left[\bar{\gamma}(x,u) + \bar{h}(x,u)\right] = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u - \mu_j\, \rho(x)u - (1-\lambda)d\, \rho(x)u - \lambda\left[\tilde{\gamma}(x,u) + \tilde{h}(x,u)\right] = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

where $\lambda \in [0,1]$.
It is clear that for $\lambda = 0$, Eq.(26) has only the trivial weak solution. Now, if u is a (weak) solution of (26) for some $\lambda \in (0,1]$, it follows from inequalities (24) that

$$(1-\lambda)d\, m(x) + \lambda\bar{\gamma}(x,u) \le (\beta+d)m(x) + d \quad \text{and} \quad (1-\lambda)d\, \rho(x) + \lambda\tilde{\gamma}(x,u) \le (\beta+d)\rho(x) + d.$$

Therefore, using Lemma 3.3, Hölder inequality, the Sobolev Embedding Theorem, and the continuity of the trace operator, one gets that

$$\begin{aligned} 0 &= (u,w)_{(c,\sigma)} - \mu_j \left(\int m(x)uw + \oint \rho(x)uw \right) - \int \bar{\tau}(x,u)w + \oint \tilde{\tau}(x,u)w \\ &\quad - \lambda \int \bar{h}(x,u)w - \lambda \oint \tilde{h}(x,u)w \\ &\ge (\delta - \tilde{k}\, d) \left\| u^{\perp} \right\|^2_{(c,\sigma)} - \bar{k} \left(\left\| u^{\perp} \right\|_{(c,\sigma)} + \|u^0\|_{(c,\sigma)} \right) \\ &\ge C_\delta \left\| u^{\perp} \right\|^2_{(c,\sigma)} - \bar{k} \left(\left\| u^{\perp} \right\|_{(c,\sigma)} + \|u^0\|_{(c,\sigma)} \right), \end{aligned}$$

where $w = \tilde{u} + v - \bar{u} - u^0$, $u^{\perp} = \tilde{u} + v + \bar{u}$, $\bar{\tau}(x,u) = (1-\lambda)d\, m(x) + \lambda\bar{\gamma}(x,u)$ and $\tilde{\tau}(x,u) = (1-\lambda)d\, \rho(x) + \lambda\tilde{\gamma}(x,u)$. For d sufficiently small, it follows that $C_\delta > 0$. Taking $a = \dfrac{\bar{k}}{2\bar{C}_\delta}$ we get that

$$\left\|u^{\perp}\right\|_{(c,\sigma)} \leq a + \left(a^2 + 2a\left\|u^0\right\|_{(c,\sigma)}\right)^{1/2}. \tag{27}$$

Step 3. We claim that there exists a constant $C > 0$ such that

$$\|u\|_{H^1} < C \tag{28}$$

for any (possible) weak solution $u \in H^1(\Omega)$ of (26) (C is independent of u and λ). If we assume that the claim does not hold, then there exist sequences (λ_n) in the interval $(0, 1]$ and (u_n) in $H^1(\Omega)$ with $\|u_n\|_{H^1} \to \infty$ such that u_n is a (weak) solution of the following problem

$$\begin{cases} -\Delta u + c(x)u - \mu_j\, m(x)u - (1-\lambda_n)d\, m(x))u - \lambda_n\, f(x,u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u - \mu_j\, \rho(x)u - (1-\lambda_n)d\, \rho(x)u - \lambda_n\, g(x,u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

That is,

$$0 = (u_n, v)_{(c,\sigma)} - \mu_j (u_n, v)_{(m,\rho)} - (1-\lambda_n)\, d\, (u_n, v)_{(m,\rho)} - \lambda_n \int f(x,u_n)v - \lambda_n \oint g(x,u_n)v \tag{30}$$

for every $v \in H^1(\Omega)$. From (27), it follows that

$$\left\|u_n^0\right\|_{(c,\sigma)} \to \infty \quad \text{and} \quad \frac{\left\|u_n^{\perp}\right\|_{(c,\sigma)}}{\left\|u_n^0\right\|_{(c,\sigma)}} \to 0. \tag{31}$$

Therefore, $\dfrac{u_n}{\|u_n^0\|_{(c,\sigma)}}$ is bounded in $H^1(\Omega)$. By the reflexivity of $H^1(\Omega)$, the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ and the compactness of the trace operator, one can assume (taking a subsequence if it is necessary) that

$$\frac{u_n}{\|u_n^0\|_{(c,\sigma)}} \rightharpoonup w \text{ in } H^1(\Omega); \quad \frac{u_n}{\|u_n^0\|_{(c,\sigma)}} \to w \text{ in } L^2(\Omega) \text{ (also in } L^2(\partial\Omega));$$

$$\frac{u_n^0}{\|u_n^0\|_{(c,\sigma)}} \to w \text{ in } L^2(\Omega) \text{ (also in } L^2(\partial\Omega)).$$

Set $v_n = \dfrac{u_n^0}{\|u_n^0\|_{(c,\sigma)}}$. Substituting v in (30) by (v_n/λ_n), and taking into account the orthogonality and the fact that $v_n \in E^0$, we get

$$0 \leq (1-\lambda_n)\lambda_n^{-1} d \left\|u_n^0\right\|_{(c,\sigma)}^{-1} \left\|u_n^0\right\|_{(m,\rho)}^2 = -\int f(x,u_n)v_n - \oint g(x,u_n)v_n \tag{32}$$

By taking the liminf as $n \to \infty$, we have that

$$\liminf_{n\to\infty} \left(\int f(x,u_n)v_n + \oint g(x,u_n)v_n\right) \leq 0. \tag{33}$$

Let $I^+ = \{x \in \Omega : w(x) > 0\}$ and $I^- = \{x \in \Omega : w(x) < 0\}$. Then for a.e. $x \in I^+$ there exists $\nu(x) \in \mathbb{N}$ such that for all $n \geq \nu(x)$, one has (passing to a subsequence if necessary),

$$\left|u_n^{\perp}(x)\right|\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} < \frac{1}{4}\, w(x)$$

and

$$\left|u_n^0(x)\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} - w(x)\right| < \frac{1}{4}\, w(x).$$

Therefore, for all $n \geq \nu(x)$ one has

$$u_n(x)\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} = (u_n^0(x)+u_n^{\perp}(x))(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \geq (u_n^0(x)-|u_n^{\perp}(x)|)(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \geq \frac{1}{2}\, w(x).$$

It follows that for a.e. $x \in I^+$, there exists an integer $\nu(x) \in \mathbb{N}$ such that for all $n \geq \nu(x)$,

$$v_n(x) > 0 \quad \text{and} \quad u_n(x) \geq \frac{1}{2}\, w(x)(\left\|u_n^0\right\|_{(c,\sigma)}) \to \infty \quad (\text{since } \left\|u_n^0\right\|_{(c,\sigma)}) \to \infty).$$

On the other hand, for a.e. $x \in I^-$ there exists $\vartheta(x) \in \mathbb{N}$ such that for all $n \geq \vartheta(x)$,

$$u_n(x)\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} = (u_n^0(x)+u_n^{\perp}(x))(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \leq (u_n^0(x)+|u_n^{\perp}(x)|)(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \leq \frac{1}{2}\, w(x).$$

Therefore, for $n \geq \vartheta(x)$, $u_n(x) \leq \frac{1}{2}\, w(x)(\left\|u_n^0\right\|_{(c,\sigma)}) \to -\infty$.
In order to apply Fatou's Lemma we need to find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$f(x,u_n)\, v_n \geq \bar{l}(x) \text{ a.e.} \quad \text{and} \quad g(x,u_n)\, v_n \geq \tilde{l}(x) \text{ a.e.,} \tag{34}$$

for some $\bar{l} \in L^1(\Omega)$ and $\tilde{l} \in L^1(\partial\Omega)$. Indeed, from (27) one gets

$$\left\|u_n^{\perp}\right\|_{(c,\sigma)}^2 \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \leq 2a\left\|u_n^{\perp}\right\|_{(c,\sigma)} \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} + 2a.$$

Therefore, by (31) one has that for $n \geq n_0$, $\left\|u_n^{\perp}\right\|_{(c,\sigma)}^2 \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \leq C$, where C is a constant independent of n. Since $\bar{\gamma}(x,u_n(x)) \geq 0$, one has that for $n \geq n_0$,

$$\begin{aligned}
\bar{\gamma}(x,u_n(x))u_n(x)v_n(x) &= \bar{\gamma}(x,u_n(x))u_n(x)u_n^0(x)\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \\
&= \frac{1}{2}\bar{\gamma}(x,u_n(x))\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1}[(u_n(x))^2 + (u_n^0(x))^2 - (u_n(x)-u_n^0(x))^2] \\
&\geq -\frac{1}{2}\bar{\gamma}(x,u_n(x))(u_n^{\perp}(x))^2\left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \geq -\bar{C}\bar{\gamma}(x,u_n(x))\, l_1(x),
\end{aligned}$$

where $l_1 \in L^1(\Omega)$, and $\bar{C} > 0$ are independent of n. Therefore, for $n \geq n_0$,

$$\bar{\gamma}(x,u_n(x))u_n(x)v_n(x) \geq -\bar{C}(\beta\, m(x) + d)\, l_1(x).$$

Now, using the decomposition of f, one has that for $n \geq n_0$,

$$\begin{aligned}
f(x,u_n(x))v_n(x) &= \bar{\gamma}(x,u_n(x))u_n(x)v_n(x) + \bar{h}(x,u_n(x))v_n(x) \\
&\geq -\bar{C}(\beta\, m(x)+d)\, l_1(x) - K_1\, l_2(x) = \bar{l}(x),
\end{aligned}$$

where $l_2 \in L^1(\Omega)$. We use similar arguments to obtain the function $\tilde{l}$ in (34). Notice that it follows from (32) and (34) that $\sup \int f(x,u_n)v_n < \infty$ and $\sup \oint g(x,u_n)v_n < \infty$. Therefore, by Fatou's Lemma and the properties of lim inf, one has

$$\int_{w>0} f_+ w \leq \liminf_{n\to\infty} \int_{w>0} f(x,u_n)v_n; \quad \oint_{w>0} g_+ w \leq \liminf_{n\to\infty} \oint_{w>0} g(x,u_n)v_n$$

$$\int_{w<0} f_- w \leq \liminf_{n\to\infty} \int_{w<0} f(x,u_n)v_n; \quad \oint_{w<0} g_- w \leq \liminf_{n\to\infty} \oint_{w<0} g(x,u_n)v_n,$$

and that

$$\int_{w>0} f_+ w + \oint_{w>0} g_+ w + \int_{w<0} f_- w + \oint_{w<0} g_- w \leq 0,$$

which contradicts the assumption (14). Thus the claim holds.

Step 4. We use the Leray-Schauder Fixed Point Theorem combined with some duality arguments.

Define $T : H^1(\Omega) \to (H^1(\Omega))^*$ by

$$T(u)v = \int \nabla u \nabla v + \int c(x)uv + \oint \sigma(x)uv - (\mu_j + d)\left(\int m(x)uw + \oint \rho(x)uw\right).$$

It follows from Theorem 2.1 and Remark 2.2 that T is linear, continuous and bijective. Therefore, by the Open Mapping Theorem we have that T^{-1} is continuous. From (26) one sees that

$$0 = T(u)v - \lambda\left(\int (f(x,u) + d\,m(x)u)v + \oint (g(x,u) + d\,\rho(x)u)v\right)$$

where $\lambda \in [0,1]$. Applying T^{-1} we get $0 = u - \lambda[T^{-1}J'_f(u) - T^{-1}J'_g(u)]$ where $J'_f(u)v = \int (f(x,u) + d\,m(x)u)v$, $J'_g(u)v = \oint (g(x,u) + d\,\rho(x)u)v$. Now, let M be defined by

$$Mu := T^{-1}J'_f(u) - T^{-1}J'_g(u).$$

Notice that from the continuity of T^{-1} and the compactness of J'_f and J'_g (see [19]) we have that M is a compact operator from $H^1(\Omega)$ to itself. Therefore, one sees that $u - \lambda\, Mu = 0$. It follows from the a priori estimate (28) and the Leray-Schauder Fixed Point Theorem that M has a fixed point. Thus, Problem (13) has a (weak) solution. The proof is complete. □

Remark 3.6 We (briefly) indicate how some of our results extend previous ones in the literature.

(i) In [3] no reaction term f is considered and the nonlinear perturbation g is sublinear at infinity.

(ii) In [17] the p-Laplacian is considered and the nonlinear perturbations f and g are bounded.

Here, we consider the case $p = 2$ and the nonlinear perturbations f and g may be unbounded, with at most linear growth asymptotically.

Remark 3.7 If the boundary nonlinearity g is Lipschitz in u, uniformly in x and the functions $c, m \in L^\infty(\Omega)$, $\sigma, \rho \in C^1(\bar{\Omega})$, one can show with a slight modification of the proof that the solution obtained in Theorem 3.1 is actually in $H^2(\Omega)$.

Remark 3.8 Our resonance results remain valid if one considers nonlinear equations with a more general linear part (in divergence form) with variable coefficients:

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{35}$$

where $\sigma \in L^\infty(\partial\Omega)$ with $\sigma(x) \geq 0$ a.e. on $\partial\Omega$, and $\partial/\partial\nu := \nu \cdot A\nabla$ is the (unit) outer conormal derivative. The matrix $A(x) := \big(a_{ij}(x)\big)$ is symmetric with $a_{ij} \in L^\infty(\Omega)$ such that there is a constant $\gamma > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\langle A(x)\xi, \xi\rangle \geq \gamma|\xi|^2 \quad \text{a.e. on } \Omega.$$

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